Generalizing the Laplace Operator to a Riemannian Manifold

Quinn Mueller

April 2015

Note:

I have modified this paper slightly from its original form so that it should be approachable with a basic understanding of ordinary differential equations.

1 Introduction

In this paper I will first discuss the Riemannian manifold and its relation to Euclidean space. From there I will discuss the generalizations of the gradient of a scalar function and the divergence of a vector field within a Riemannian manifold. With these two generalizations will then be used along with the identity $\nabla^2 u = \operatorname{div} \operatorname{grad} u$ to arrive at the generalized Laplace operator. In this context the generalized operator is sometimes referred to as the Laplace-Beltrami operator, although I will simply refer to it as the Laplace operator.

2 The Riemannian Manifold

Before discussing a Riemannian manifold, we need to define a few concepts that generalize from Euclidean space.

2.1 Metrics

In Euclidean space be have the idea of distance between two points x, y as the Euclidean distance, $d(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$. The idea of distance can be generalized to any set, and is called a metric on that set. We can define a metric formally as follows.[2]

Definition. For a non-empty set X, a function $d : X \times X \to \mathbb{R}$ is said to be a *metric* on X if it satisfies the following:

(i)	$d(x, y) \geq 0$	$\forall x, y \in X$
(ii)	$d(x, y) = 0 \Leftrightarrow x = y$	$x, y \in X$
(iii)	$d(x, y) = d(y, x)$	$\forall x, y \in X$
(iv)	$d(x, y) \leq d(x, z) + d(z, y)$	$\forall x, y, z \in X$

2.2 Metric Tensors

Besides distance we also need the notion of directions, and displacements in those directions, so we will consider a subset of metrics called metric tensors. The metric tensor does not directly define the distance function, but rather the inner product on our space. For j, k in our basis, I will be using the following notation:

$$\begin{aligned} g_{jk} &= \text{ jk component of the metric tensor} \\ (g_{jk}) &= \text{ the component matrix} \\ g &= \det(g_{jk}) \\ g^{jk} &= \text{ jk component of the inverse metric} \end{aligned}$$

2.3 A Riemannian Manifold

The last object we need to define is the manifold.

Definition. An *n-dimensional manifold* is a topological space that is locally homeomorphic to Euclidean space $\mathbb{R}^n$.[7]

This means that locally a manifold can be stretched in such a way that it resembles Euclidean space. We can now define a Riemannian manifold as follows.

Definition. A *Riemannian manifold* is a smooth, differentiable manifold with an associated positive-definite symmetric metric tensor.[1][7]

If a Riemannian manifold is complete (it has no holes), we can define our distance function as the arc length of the shortest curve connecting two points.[4] Since the metric tensor for a Riemannian manifold must be positive-definite, we know that: the determinant g is positive, and the inverse metric (g^{jk}) exists. For the simple case of a two dimensional Riemannian manifold, the standard notation for a basis (u, v) is:

$$(g_{jk}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{2.1}$$

Where $EG - F^2 > 0$.

3 Deriving the Laplace Operator

Now that we've defined a Riemannian manifold and requisite manifold, we can now look at how the Laplace operator on Euclidean space generalizes to the Laplace operator on a general Riemannian manifold. While it is possible to more rigorously derive this generalization from the properties of the Laplace operator itself, I frankly don't understand enough of the math behind such a derivation, so here we will use the identity from vector calculus that:

$$\nabla^2(u) = \text{div}(\text{grad } u) \tag{3.1}$$

3.1 Gradient

To do this we must first generalize the gradient. Since our basis vectors are not unit length, we must scale the directional derivative accordingly. From what I can discern, the explicit gradient function may depend on whether or not it is expressed in terms of the basis vectors or covectors of the space. Regardless, in terms of the Riemannian metric, the j^{th} component of the gradient for $X = \text{grad}(u)$ is given by:[5]

$$X^j = \sum_{k=1}^{n} g^{jk} \frac{\partial u}{\partial x_k} \tag{3.2}$$

Which in our standard two dimensional (x, y) basis, looks like:

$$\text{grad}\, u = \left\{ g^{xx} \frac{\partial u}{\partial x} + g^{xy} \frac{\partial u}{\partial y},\ g^{yy} \frac{\partial u}{\partial y} + g^{yx} \frac{\partial u}{\partial x} \right\}$$

3.2 Divergence

Now to generalize the divergence, the idea that divergence describes the flux of a vector field through a volume can be generalized to the change in the volume density of our manifold along the flow of our vector field[6]. As such, it depends both on the magnitude of the vector field at a point, and the value of the volume element for our manifold at that point. For an n-dimensional Riemannian manifold, the volume element is given by the n-form:

$$\omega = \sqrt{g}\, dx_1 \wedge \ldots \wedge dx_n \tag{3.3}$$

Where $\wedge$ is the wedge, or exterior, product of our basis. ω is often written as dA in two dimensions, or dV in three.

If we note that the change in the volume along our vector field X can be given by the Lie derivative of the volume element, we can equate the Lie derivative and the divergence as:

$$(\text{div}\, X)\omega = \pounds_X \omega \tag{3.4}$$

I don't understand enough of the Lie derivative to give an explicit derivation, as shown in Taylor[6], however since the Lie derivative of our n-form volume element is itself an n-form, we can divide by ω to get our scalar valued divergence as:

$$\operatorname{div} X = \frac{\pounds_X \omega}{\omega} \tag{3.5}$$

Which in terms of partial derivatives comes out to be:

$$\operatorname{div}(X) = \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g}\, X^j \right) \tag{3.6}$$

Where the $\sqrt{g}$ inside the partial derivative comes from our Lie derivative, and the $1/\sqrt{g}$ outside comes from dividing by the volume element to get volume density.

3.3 Laplace Operator

Using our identity (3.1), along with our generalized gradient (3.2), and divergence (3.6), we arrive at the following formula for our generalized Laplace operator, for a scalar valued function u.

$$\nabla^2 u = \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial x_j} \left(g^{jk} \sqrt{g}\, \frac{\partial u}{\partial x_k} \right) \tag{3.7}$$

If we write this explicitly for our two dimensional metric given in (2.1), we get the following giant mess.

$$\begin{aligned}
&\nabla^2 u(x,y) = \\
&\frac{1}{\sqrt{EG-F^2}} \frac{\partial}{\partial x} \left(\frac{G}{EG-F^2} \sqrt{EG-F^2}\, \frac{\partial}{\partial x} \left(u(x,y)\right) \right) + \\
&\frac{1}{\sqrt{EG-F^2}} \frac{\partial}{\partial x} \left(\frac{-F}{EG-F^2} \sqrt{EG-F^2}\, \frac{\partial}{\partial y} \left(u(x,y)\right) \right) + \\
&\frac{1}{\sqrt{EG-F^2}} \frac{\partial}{\partial y} \left(\frac{-F}{EG-F^2} \sqrt{EG-F^2}\, \frac{\partial}{\partial x} \left(u(x,y)\right) \right) + \\
&\frac{1}{\sqrt{EG-F^2}} \frac{\partial}{\partial y} \left(\frac{E}{EG-F^2} \sqrt{EG-F^2}\, \frac{\partial}{\partial y} \left(u(x,y)\right) \right)
\end{aligned}$$

If our metric is orthogonal, recalling that the metric tensor defines the inner product, we know that $F = 0$, and we get the more pleasant Laplace operator.

$$\begin{aligned}\nabla^2 u(x,y) =& \frac{1}{\sqrt{EG}} \frac{\partial}{\partial x}\left(\frac{\sqrt{EG}}{E}\frac{\partial}{\partial x}\left(u(x,y)\right)\right) + \\ & \frac{1}{\sqrt{EG}} \frac{\partial}{\partial y}\left(\frac{\sqrt{EG}}{G}\frac{\partial}{\partial y}\left(u(x,y)\right)\right)\end{aligned} \tag{3.8}$$

3.4 Deriving the Laplace Operator in Polar Coordinates

Polar coordinates can be shown to be a metric on Euclidean space, the simplest of Riemannian manifolds, given by:[4]

$$(g_{jk}) = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{3.9}$$

For the basis (r, θ). (A derivation of this metric can be done using (4.2)). Substituting these values into (3.8) and simplifying, we get the expected Laplace operator for polar coordinates.

$$\begin{aligned}\nabla^2 u(r,\theta) =& \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u(r,\theta)\right) + \\ & \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}u(r,\theta)\end{aligned}$$

4 Example on the Unit 2-Sphere

Here we will look at how a Riemannian manifold can be used to describe a 2-sphere, and how that description can be used to solve the heat equation on the 2-sphere. When embedded in $\mathbb{R}^3$ the 2-Sphere can be parametrized in terms of (x, y) as:

$$\begin{aligned}&\mathbf{S}(x,y) = \{\sin x \sin y, \sin x \cos y, \cos x\} \\ &0 \le x \le \pi, \;\; 0 \le y \le 2\pi\end{aligned} \tag{4.1}$$

For which we can compute the induced metric by:

$$\begin{aligned}E &= \mathbf{S_x} \cdot \mathbf{S_x} = 1 \\ F &= \mathbf{S_x} \cdot \mathbf{S_y} = 0 \\ G &= \mathbf{S_y} \cdot \mathbf{S_y} = \sin^2 x\end{aligned} \tag{4.2}$$

When we substitute these values for E, F and G into the Laplace operator for an orthogonal metric (3.8) we get:

$$\nabla^2 u(x,y) = \frac{1}{\sin x}\frac{\partial}{\partial x}\left(\sin x \frac{\partial u}{\partial x}\right) + \frac{1}{\sin x}\frac{\partial}{\partial y}\left(\frac{1}{\sin x}\frac{\partial u}{\partial y}\right) \tag{4.3}$$

Which expands to:

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\cos x}{\sin x}\frac{\partial u}{\partial x} + \frac{1}{\sin^2 x}\frac{\partial^2 u}{\partial y^2} \tag{4.4}$$

4.1 Heat Equation

Here we will solve the heat equation on the full unit sphere with a general initial condition.

$$\nabla^2 u = \frac{\partial u}{\partial t} \tag{4.5}$$

$$IC : u(0,x,y) = f(x,y) \tag{4.6}$$

The boundary conditions are implied by the periodic nature of our parametrization as:

$$u(t,x,0) = u(t,x,2\pi) \tag{4.7}$$

$$u_y(t,x,0) = u_y(t,x,2\pi) \tag{4.8}$$

$$|u(t,0,y)| < \infty \tag{4.9}$$

$$|u(t,\pi,y)| < \infty \tag{4.10}$$

If we make the separation of variables assumption $u(t,x,y) = T(t)X(x)Y(y)$, (4.5) becomes:

$$T'XY = TX''Y + \frac{\cos x}{\sin x}TX'Y + \frac{1}{\sin^2 x}TXY'' \tag{4.11}$$

Which we can separate for t as:

$$\frac{T'}{T} = \frac{X''}{X} + \frac{\cos x}{\sin x}\frac{X'}{X} + \frac{1}{\sin^2 x}\frac{Y''}{Y} = -\lambda \tag{4.12}$$

We can further separate for y as:

$$\frac{Y''}{Y} = -\sin^2 x\frac{X''}{X} - \sin x\cos x\frac{X'}{X} - \lambda\sin^2 x = -\sigma \tag{4.13}$$

This lends us to solve the following ODE in y:

$$Y'' + \sigma Y = 0 \tag{4.14}$$
$$Y(0) = Y(2\pi) \tag{4.15}$$
$$Y'(0) = Y'(2\pi) \tag{4.16}$$

Where the boundary conditions come from (4.7) and (4.8). This standard ODE has solutions:

$$Y_n = A_n \sin ny + B_n \cos ny \tag{4.17}$$

For:

$$\sigma = n^2 \tag{4.18}$$
$$n = 0, 1, 2, \dots \tag{4.19}$$

If we substitute (4.18) into (4.13), we get the following ODE in x.

$$\sin^2 x X'' + \sin x \cos x X' + \left(\lambda \sin^2 x - n^2\right) X = 0 \tag{4.20}$$

We can divide by $\sin^2 x$ to get:

$$X'' + \frac{\cos x}{\sin x} X' + \left(\lambda - \frac{n^2}{\sin^2 x}\right) X = 0 \tag{4.21}$$

To solve this equation, we need to look at the associated Legendre polynomials (commonly denoted P_a^b), which solve the differential equation:

$$\left(1 - z^2\right) \frac{d^2 f}{dz^2} - 2z \frac{df}{dz} + \left(a(a+1) - \frac{b^2}{1 - z^2}\right) f = 0 \tag{4.22}$$

For a positive integer a and b a non-negative integer such that $b \leq a$. If we let $z = \cos x$ the associated Legendre differential equation becomes:

$$\frac{d^2 f}{dx^2} + \frac{\cos x}{\sin x} \frac{df}{dx} + \left(a(a+1) - \frac{b^2}{\sin^2 x}\right) f = 0 \tag{4.23}$$

Which looks like our ODE (4.21) if we let $\lambda = m(m+1)$ for a positive integer m. So, along with the associated Legendre functions of the second kind, Q_a^b, we get the solution for X.

$$X_m = C_m P_m^n(\cos x) + D_m Q_m^n(\cos x) \tag{4.24}$$

Imposing our boundary condition (4.9), (4.10) such that X is finite at $x = 0, x = \pi$, we know $D_m = 0$ as $Q_m^n(z)$ is infinite for $z = 1, z = -1$. This gives us our final solution for X_m.

$$X_m = C_m P_m^n(\cos x) \tag{4.25}$$

Going back to our ODE in t from (4.12), we have:

$$T' + m(m+1)T = 0 \tag{4.26}$$

Which has solutions:

$$T_m = E_m e^{-m(m+1)t} \tag{4.27}$$

When we substitute (4.17), (4.25), (4.27) into our separation of variables assumption we get our solution for $u(t,x,y)$.

$$u(t,x,y) = \sum_{m=1}^{\infty}\sum_{n=0}^{m}\left[e^{-m(m+1)t}\left(A_{mn}\sin nyP_m^n(\cos x) + B_{mn}\cos nyP_m^n(\cos x)\right)\right] \tag{4.28}$$

With our initial condition (4.6) we can solve for the constants A_{mn} and B_{mn} using the orthogonality of sines and cosines, and the following orthogonality of associated Legendre polynomials.

$$\int_{-1}^{1} P_a^b(z)P_c^b(z)dz = \begin{cases} \frac{2}{2a+1}\frac{(a+b)!}{(a-b)!} & \text{if } a=c \\ 0 & \text{otherwise} \end{cases} \tag{4.29}$$

Which for $z = \cos x$ becomes the weighted orthogonality condition:

$$\int_{0}^{\pi} P_a^b(\cos x)P_c^b(\cos x)\sin x dx = \begin{cases} \frac{2}{2a+1}\frac{(a+b)!}{(a-b)!} & \text{if } a=c \\ 0 & \text{otherwise} \end{cases} \tag{4.30}$$

So:

$$A_{mn} = \frac{(2m+1)(m-n)!}{2(m-n)!}\frac{1}{\pi}\int_0^{\pi}\int_0^{2\pi} f(x,y)\sin nyP_m^n(\cos x)\sin xdydx \tag{4.31}$$

$$B_{mn} = \frac{(2m+1)(m-n)!}{2(m-n)!}\frac{1}{\pi}\int_0^{\pi}\int_0^{2\pi} f(x,y)\cos nyP_m^n(\cos x)\sin xdydx \tag{4.32}$$

And we have our final solution to the heat equation on the full unit sphere, given an initial temperature distribution.

References

[1] Coquereaux, Robert, and Arkadiusz Jadczyk. *Riemannian Geometry Fiber Bundles Kaluza-Klein Theories and All That.* Singapore: World Scientific Publishing, 1988. Print.

[2] Jain, Pawan K. and Khalil Ahmad. *Metric Spaces.* 2nd Ed. Pangbourne, U.K.: Alpha Sciences International, 1993. Print.

[3] Kumaresan, S. *Topology of Metric Spaces.* Harrow, U.K.: Alpha Science International, 2005. Print.

[4] Oprea, John. *Differential Geometry and Its Applications.* 2nd Ed. Washington: The Mathematical Association of America, 2007. Print.

[5] Sogge, Christopher D. *Hangzhou Lectures on Eigenfunctions of the Laplacian.* Princeton: Princeton University Press, 2014. Print.

[6] Taylor, Michael E. *Partial Differential Equations I: Basic Theory.* 2nd Ed. New York: Springer. E-Book.

[7] Walschap, Gerard. *Metric Structures in Differential Geometry.* New York: Springer, 2004. Print.